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Extension of Goldstein's series for the Oseen drag of a sphere

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Goldstein's expansion of the Oseen drag of a sphere in powers of Reynolds number is extended to 24 terms by computer. The convergence is found to be limited by a simple pole at $R = -4{\cdot}18172$. The series is recast using an Euler transformation and other devices to yield accurate results for large R.

1. Introduction

In 1929 Goldstein published an analysis of the drag of a sphere according to the Oseen linearization of the Navier–Stokes equations, and gave the expansion

$$C_D \equiv \frac{D}{\pi\rho U^2 a^2} = \frac{12}{R}\left(1+\frac{3}{16}R-\frac{19}{1280}R^2+\frac{71}{20{,}480}R^3-\frac{30{,}179}{34{,}406{,}400}R^4 + \frac{122{,}519}{550{,}502{,}400}R^5-\dots\right). \tag{1.1}$$

Here $R = 2Ua/\nu$ is the Reynolds number based upon diameter. We have corrected the last denominator (from 560,742,400) according to Shanks (1955). The first term was given by Stokes in 1851, and the second by Oseen in 1910. Goldstein remarks that his approximation is useful only up to $R = 2$.

Later Goldstein (1938) pointed out that the corresponding expansion for the full Navier–Stokes equations will differ from (1.1) after the first two terms. Proudman & Pearson (1957) showed that it has a more complicated structure involving logarithms as well as powers of R, beginning with $R^2 \ln R$. This might suggest that extension of Goldstein's series to higher powers of R would be profitless. However, it is likely that the Oseen model will continue to provide insight and guidance for the Navier–Stokes equations, especially for separated flows. Questions of the radius of convergence, the analytical structure of the function represented by the series, and effective means of recasting it to improve accuracy—all of which arise also in the Navier–Stokes problem—can be clarified by adding terms to Goldstein's series.

Hand calculation of more than one or two additional coefficients would seem to be humanly impossible; but the routine operations involved can now be delegated to a computer. Machine extension of perturbation series is less developed in fluid mechanics than in such fields as celestial mechanics (see, for example, Deprit & Rom 1967). Coupled with the application of transformations to improve the accuracy of the resulting series, this new role for the computer provides a promising technique for attacking a variety of problems in fluid mechanics, of which the present analysis provides a simple example.

2. Machine solution

A computer program was written in the FORTRAN IV language to calculate any given number of the coefficients in Goldstein's series (1.1). It consists of some 300 statements and (in Goldstein's notation) computes in succession

(1) the coefficients in the sums (36) and (37) for χ_m and ψ_n from the second of Goldstein's equations (32),
(2) the coefficients in χ'_n from the first of (32),
(3) the coefficients in $\Psi_{n,m}$ from (40) and (23),
(4) the coefficients in $\Psi'_{n,m}$ from (39),
(5) the coefficients in $X_{n,m}$ from (47) and (51),
(6) the coefficients of the expansion of $\lambda_{n,m}$ in powers of ξ from (46),
(7) the coefficients of the expansion of B_m in powers of ξ from (18),
(8) the coefficients of the expansion of k_D in powers of ξ from (57).

The actual program involves various subtleties not suggested by this bare outline.

An effort to calculate the coefficients as rational numbers using integer arithmetic had to be abandoned in favour of floating-point arithmetic because the maximum allowed integer is soon exceeded. The results are consequently subject to accumulated round-off error. The accuracy was controlled only by comparing single- and double-precision calculations.

The computations were carried out on the Stanford IBM 360/67 computer. The storage of the machine was exhausted with the calculation of 24 terms of the series, and this number was regarded as sufficient, so that no effort was made to optimize or augment the storage. The double-precision calculation of 24 terms required just 1 min.

The direct product of Goldstein's analysis is a series in powers of his $\xi_0 = Ua/2\nu$, one-quarter of the Reynolds number based on diameter, which is a parameter that appears naturally in Oseen theory. We choose to retain that natural form, because the coefficients then all have magnitude of order unity. Thus we expand the drag coefficient as

$$C_D \equiv \frac{D}{\pi\rho U^2 a^2} = \frac{12}{R}\sum_{n=0} c_n \left(\frac{R}{4}\right)^n. \tag{2.1}$$

Table 1 gives the first 24 coefficients c_n, of which the first five agree with Goldstein's values and the sixth with Shanks's correction. Comparison with the single-precision calculation suggests that these are all correct to within one unit in the eighth figure.

The coefficients alternate regularly in sign, which permits the accuracy of a finite sum to be assessed. Our 24 terms yield four significant figures ($C_D = 5{\cdot}929$) at $R = 3$, but only one figure ($C_D = 5$) at $R = 4$ and none at $R = 5$.

The alternation of signs indicates that the nearest singularity in the complex plane of Reynolds number lies on the negative real axis. [This refutes the implication of Montroll (1969) that convergence is limited by the critical Reynolds number $R_c \approx 5 \times 10^5$.] Examining the original six coefficients, the writer previously suggested (Van Dyke 1964*a*, p. 205–210) a singularity at $R = -4$. To improve this estimate we exploit not only our 18 additional coefficients but also a graphical procedure due to Domb & Sykes (1957). In helping to estimate the radius of convergence, their plot also predicts the nature of the nearest singularity.

The radius of convergence (in terms of $\frac{1}{4}R$) is, according to D'Alembert's ratio test, the limit as n becomes infinite of c_{n-1}/c_n. The inverse ratios c_n/c_{n-1} are plotted *versus* $1/n$ in figure 1. This is the Domb–Sykes plot, which (in addition to bringing

n	c_n	d_n	n	c_n	d_n
0	1·0000 0000	1·0000 0000	12	−0·1806 4216	−0·0037 1456
1	0·7500 0000	−0·2159 2879	13	0·1725 8708	−0·0032 2957
2	−0·2375 0000	−0·2595 6857	14	−0·1649 9175	−0·0027 8333
3	0·2218 7500	−0·0060 6090	15	0·1577 9401	−0·0024 1401
4	−0·2245 4613	−0·0207 6791	16	−0·1509 4019	−0·0021 3118
5	0·2278 9993	−0·0191 0239	17	0·1443 9312	−0·0019 2188
6	−0·2275 7591	−0·0135 6963	18	−0·1381 2931	−0·0017 6358
7	0·2232 5498	−0·0090 7563	19	0·1321 3363	−0·0016 3473
8	−0·2160 6031	−0·0065 0353	20	−0·1263 9492	−0·0015 2016
9	0·2073 5255	−0·0052 7441	21	0·1209 0337	−0·0014 1204
10	−0.1981 8934	−0·0046 5395	22	−0·1156 4943	−0·0013 0829
11	0.1891 9541	−0·0041 8897	23	0·1106 2351	−0·0012 1005

TABLE 1. Coefficients in Goldstein's series (2.1) and its Euler transform (4.1)

the extrapolation to the origin) has the advantage that for certain common singular functions it gives a linear variation: if

$$f(z) = \sum^{\infty} c_n z^n = \begin{cases} C(z_0 \pm z)^\alpha, & \alpha \neq 0, 1, \ldots, \\ C(z_0 \pm z)^\alpha \ln (z_0 \pm z), & \alpha = 0, 1, \ldots, \end{cases} \tag{2.2a}$$

then

$$\frac{c_n}{c_{n-1}} = \mp \frac{1}{z_0}\left(1 - \frac{1+\alpha}{n}\right). \tag{2.2b}$$

If the plot tends to become straight, its limiting slope accordingly indicates the nature of the nearest singularity. Figure 1 shows rapidly damped oscillations about a limiting horizontal line. Thus the nearest singularity is a simple pole ($\alpha = -1$).

The inset in figure 1 shows that maxima and minima appear in the plot regularly at every sixth coefficient; and the amplitude decreases by a factor of about 30 in each half cycle. Considering these facts, we estimate the intercept at $-0{\cdot}956545$. Thus Goldstein's series converges for Reynolds numbers less than

$$R_0 = 4{\cdot}18172\ldots. \tag{2.3}$$

This has not been recognized as a known transcendental. (It is close to

$$4\pi/3 = 4{\cdot}18879\ldots,$$

but not close enough.)

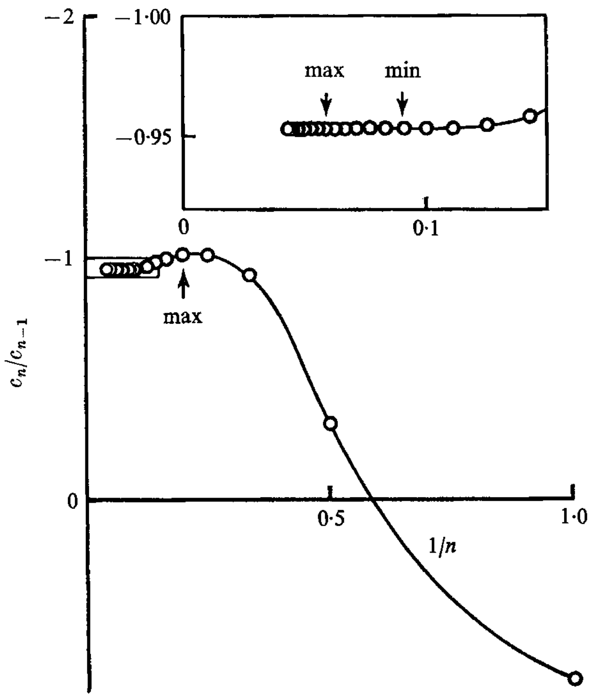

FIGURE 1. Domb–Sykes plot for Goldstein's series.

3. Analytic continuation

Although the series has a modest radius of convergence, its first 24 terms contain a great deal of information about the drag coefficient for any Reynolds number, which only awaits unveiling. (Of course this assertion rests on our expectation that a unifying physical fabric underlies the expansion.)

The pole on the negative axis of R spoils the utility of the series on most of the positive real axis, which is doubtless free of singularities. The series can therefore be improved in principle by analytic continuation into the right half-plane.

Euler transformation

One practical method of analytic continuation when only a finite number of terms are known, and the radius of convergence can be estimated, is to banish the offending singularity to infinity with a linear fractional transformation. Thus we make an Euler transformation, recasting the series in powers of the variable $R/(R_0+R)$ instead of R. This gives

$$C_D = \frac{12}{R_0}\left(\frac{R}{R_0+R}\right)^{-1} \sum_{n=0} d_n \left(\frac{R}{R_0+R}\right)^n. \tag{3.1}$$

Table 1 gives the first 24 coefficients d_n. They diminish smoothly in magnitude beyond the fourth. [We could have extracted a power of R before recasting, as was done for the skin friction on a parabola (Van Dyke 1964*b*). In not doing so we have tacitly exploited our knowledge that C_D is finite at $R = \infty$.]

All coefficients but the first are negative, so that the series now yields only an upper bound for the drag, and the accuracy is not certain. However, the values listed in the second column of table 2 are believed, on the basis of examination of successive terms, to be accurate to within one unit in the last figure given (and this is borne out by more accurate results discussed later). $R = 7{\cdot}605$ is the value at which, according to the numerical solutions of Bourot (1969), a closed recirculating ring eddy appears at the rear of the sphere.

R	Euler transform (3.1)	Rational fractions (3.4)	Euler transform plus rational fractions	Euler transform plus completion of series
4	4·86993	4·86993	4·86993	4·86993
7·605	3·30639	3·30639	3·30639	3·30639
10	2·86760	2·86760	2·86760	2·86760
20	2·122	2·12188	2·12188	2·12187
30	1·847	1·8460	1·84604	1·84603
40	1·70	1·697	1·6975	1·6974
50	1·61	1·602	1·6029	1·6028
100	1·4	1·4	1·393	1·3923
400	1·2	1·2	1·20	1·188
∞	1·1	1·1	1·1	1·06

TABLE 2. Drag coefficient from various schemes for improving Goldstein's series

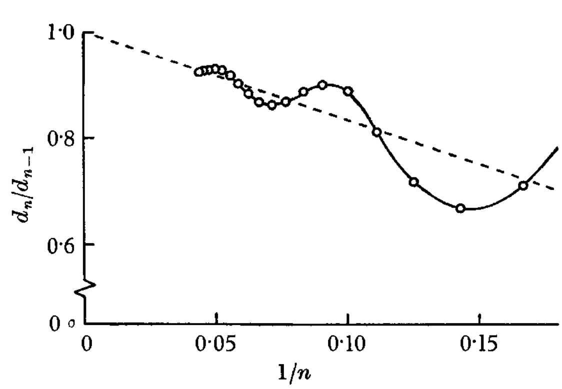

FIGURE 2. End of Domb–Sykes plot for Euler transform of Goldstein's series: – – – – –, $[1-R/(R_0+R)]^{\frac{2}{3}}$.

Because the signs are unchanging, the nearest singularity now lies on the positive real axis. Figure 2 shows the end of the Domb–Sykes plot. The damped oscillation that was invisible in figure 1 has been enormously magnified, and speeded up to an extremum every fourth coefficient. Nevertheless, it seems clear that the intercept is unity. The radius of convergence is therefore unity for $R/(R_0+R)$, and hence infinite for R itself. The limiting slope surely corresponds to some positive α in (2.2), which means that the series converges at $R = \infty$.

The convergence is slow, however, the last five partial sums giving

$$C_D = \ldots 1{\cdot}1929,\ 1{\cdot}1886,\ 1{\cdot}1845,\ 1{\cdot}1808,\ 1{\cdot}1773 \quad \text{at} \quad R = \infty. \tag{3.2}$$

Applying the non-linear transformation of Shanks (1955) to the last three values (which amounts to extrapolating on the assumption that they are part of a geometric series) yields 1·13. Comparison at lower R suggests that this artifice gives again an upper bound, and we are able to retain with confidence only two significant figures in table 2. Thus we are far from being able to reproduce the value 1·06 calculated numerically by Stewartson (1956). [In unpublished work J.-M. Bourot has refined this value to $C_D = 1{\cdot}05825$ using 28 terms of Stewartson's expansion.]

The oscillation complicates the estimation of the limiting slope. However, Stewartson has pointed out to the writer that Roberts's (1967) analysis of the analogous problem of the Hartmann layer in a circular pipe implies that here

$$C_D \sim C_{D\infty} + KR^{-\frac{2}{3}} + \ldots \quad \text{as} \quad R \to \infty. \tag{3.3}$$

The curve in figure 2 is seen to twine closely about the straight line corresponding to this behaviour.

Rational fractions

The Euler transformation does not take advantage of our knowledge of the nature of the nearest singularity. A method that does is to multiply out that singularity, and recast the remainder as a rational fraction. Rational fractions (or Padé approximants) possess remarkable properties of analytical continuation that, although not fully understood, prove valuable in many physical applications (Baker 1965).

Here, taking advantage of the extra information that the drag coefficient is finite at infinite Reynolds number, we choose to form (from an odd number of terms of the series) rational fractions whose numerators are of one degree higher than the denominators. The result will then have a finite limit when the pole is reintroduced. For example, three terms give

$$C_D \approx 12\,\frac{R_0 + (1+\frac{3}{16}R_0) + (\frac{3}{16} - \frac{19}{1280}R_0)\,R^2}{R(R_0+R)}. \tag{3.4}$$

In higher approximations, this procedure is conveniently carried out on the computer using the 'epsilon algorithm' of Wynn (1956). It is inapplicable at $R = \infty$, however, and must be replaced by a procedure involving determinants.

In contrast to the smooth monotonic variation of successive Euler transforms, rational fractions oscillate in an erratic and unpredictable way (as they do in other physical problems). Nevertheless, the magnitude of the oscillations decreases so rapidly with the number of terms that much greater accuracy is achieved than with the Euler transformation. Thus in the third column of table 2 we have been able to add at least one more secure significant figure for R up to 50.

Combined procedure

The idea suggests itself of combining the smoothness of the Euler transformation with the convergence acceleration of rational fractions. Thus from the even partial sums of the Euler transform (3·1) we have formed rational fractions with numerator and denominator of equal degree [the first being just two terms of (3.1) itself]. It is clear from internal evidence that the accuracy is greatly increased. Thus in the fourth column of table 2 we have been able to add at least one more significant figure at each Reynolds number except $R = \infty$.

Our values now agree completely with the six-figure numerical results of Bourot (1969) at $R = 4$, 10, 20, and 30, and with his revised (unpublished) value at $R = 7{\cdot}605$, and agree to five figures with his unpublished values $C_D = 1{\cdot}69742$ at $R = 40$ and $1{\cdot}60279$ at $R = 50$.

Completion of the series

Despite their effectiveness, rational fractions are suspect because they must be applied blindly. A more logical way of refining the result of the Euler transformation is to complete the series as suggested by (3.3) and figure 2. Thus the remainder for the sum in (3.1) is approximated by that for the expansion of

$$C_n[1-R/(R_0+R)]^{\frac{2}{3}},$$

where C_n is chosen to make the nth terms agree. The consistency of the results for various $n \leqslant 23$ suggests that this device still further increases the accuracy, giving the values shown in the last column of table 2. In particular, this has at last permitted us to reproduce Stewartson's value of $1{\cdot}06$ at $R = \infty$.

4. Discussion

Whereas the Oseen drag is only marginally described by the original six terms of Goldstein's series, our extension to 24 terms provides a nearly complete characterization of the function in the entire complex plane of Reynolds number. Thus we have found the location and nature of the singularities, and by judicious recasting have been able to extract five-figure accuracy at Reynolds numbers up to 25 times as great as the original radius of convergence, and three-figure accuracy even at $R = \infty$.

In the analogous expansion for the Navier–Stokes equations, the calculation of additional terms unfortunately cannot be delegated to the computer. Chester & Breach (1969) have, by detailed analysis, completed the third approximation and found part of the fourth. Comparing with the experiments of Maxworthy (1965), they conclude that their result is useful only up to $R = 0{\cdot}5$. Improvement by recasting would require more sophisticated schemes than those used here, which are appropriate only for power series. Proudman (1969) has proposed one method that yields promising results.

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